Non-Abelian Duality and Canonical Transformations

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Abstract

We construct explicit canonical transformations producing non-abelian duals in principal chiral models with arbitrary group G. Some comments concerning the extension to more general σ-models, like WZW models, are given.

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1 Introduction

Recently some progress has been made in the understanding of target space duality as a symmetry of string theory in general backgrounds. Some general review references are [1]. These developments started with the work of Buscher [2] where the dual theory of an arbitrary σ-model with an abelian isometry was constructed. This was elaborated further in [3], where an alternative method of constructing the dual was presented. This method has the advantage that it can be easily generalized to σ-models with non-abelian isometries, as was first done in [6]. Basically it consists in gauging the isometry with some gauge fields constrained to be flat by means of some Lagrange multipliers. Integrating out these Lagrange multipliers and fixing the gauge the initial model is recovered. If instead one integrates first the gauge fields and then fixes the gauge the dual model, in which the Lagrange multipliers appear as new coordinates, appears. Further works along these lines are for instance [4, 5] for the abelian case and [7, 8] for the non-abelian one.

For abelian duality it was shown in [11] that there exists a much simpler and straightforward way of reaching the dual theory, and this is in terms of a canonical transformation[1]. The explicit canonical transformation yielding the dual theory was constructed and shown to reproduce Buscher's formulas. In this way duality can be understood as a subgroup of the whole group of canonical transformations in the phase space of the theory. A key point in the construction of the canonical transformation is the fact that there exists a system of coordinates adapted to the isometry, i.e. such that only its derivatives appear in the Lagrangian. This turns out to be crucial in order to obtain a local dual Hamiltonian since it allows to express its dependence on the dual variables and conjugate canonical momenta as local functions of the initial variables and momenta. Plenty of useful information about the dual model can be obtained very easily within this approach, as shown in [11]. For instance, the extension to arbitrary genus surfaces of the proof of duality given in [2, 3] for spherical world-sheets and the multivaluedness and periods of the dual variables can be worked out by just considering the implementation of the canonical transformation in the path integral. Also, in the particular case of WZW-models [13] it becomes rather simple to prove that the full duality group is given by $\mathrm{Aut}(G)_L \times \mathrm{Aut}(G)_R$, where L, R refer to the left and right currents on the model with group G, and $\mathrm{Aut}(G)$ are the automorphisms of G, both inner and outer. It is clarified that this is possible because the currents are chirally conserved, this being the reason why the transformation leads to a local expression for the dual currents. If this is not the case those currents not commuting with the one used to perform duality become non-local in the dual theory and the symmetry to which they are associated is no longer manifest. This clarifies the point of the apparent loss of symmetries after duality transformations.

In view of the simplicity of this approach it would be very interesting to also have a full understanding of non-abelian duality in terms of canonical transformations. As pointed out in [11] this does not seem obvious to do, at least as a generalization of the abelian case, basically because a system of coordinates adapted to the whole set of non-commuting isometries does not exist, and as we mentioned above this is crucial in order to obtain a

[1] Canonical transformations were first used in the context of duality symmetries in [10] for constant or only time-dependent backgrounds.

local dual Hamiltonian. However, in [16] the non-abelian dual of the $SU(2)$ principal chiral model with respect to the left action of the whole group was constructed out of a canonical transformation[2]. This non-abelian model had been constructed before in the literature [5, 17] following the standard procedure described in [6]. Although a system of coordinates adapted to the isometry does not exist it is still possible to eliminate the explicit dependence on the original variables owing to some nice cancellations.

In this letter we present a generalization of this construction to principal chiral models with arbitrary group G. Although at first sight not obvious, it can be seen that the explicit dependence of the Hamiltonian on the initial variables disappears after the canonical transformation is made. The generating functional is argued to be exact to all orders in α', as in the abelian case. These general results open the possibility of understanding non-abelian duality in terms of canonical transformations. One of the nicest features of having a canonical transformation description of non-abelian duality is that the generalization to arbitrary Riemann surfaces of the result in [6] for spherical world-sheets and the multivaluedness and periods of the dual variables could be worked out by just studying the implementation of the transformation in the path integral. As it was discussed in detail in [5] the main difficulty in trying to do this within the ordinary approach of De la Ossa and Quevedo is that the Hodge decomposition and the splitting of the flat connection part of the auxiliary gauge field do not coincide. It was just this coincidence what allowed to work out this problem in the abelian case. It would be interesting as well to obtain the group under which the conserved currents transform under non-abelian duality and the explicit (non-local) expressions for the duals of the conserved currents of the initial theory. Also, the generalization to WZW-models could be very useful in determining further if the theory is self-dual, as it seems to be from the semiclassical calculations[3].

2 Non-abelian duality as a canonical transformation

Let us start this section by briefly reviewing the basic features of abelian duality in the canonical transformation approach.

Given a general σ-model with an abelian isometry represented by translations of a θ-coordinate:

$$L = \frac{1}{2} g_{00} (\dot{\theta}^2 - \theta'^{\,2}) + (\dot{\theta} + \theta') J_- + (\dot{\theta} - \theta') J_+ + V, \qquad (2.\ 1)$$

where:

$$J_- = \frac{1}{2}(g_{0i} + b_{0i}) \partial_- x^i, \qquad J_+ = \frac{1}{2}(g_{0i} - b_{0i}) \partial_+ x^i$$
$$V = \frac{1}{2}(g_{ij} + b_{ij}) \partial_+ x^i \partial_- x^j, \qquad (2.\ 2)$$

we can obtain the abelian dual with respect to this isometry by performing the canonical transformation [11]:

$$p_\theta = -\tilde{\theta}', \qquad p_{\tilde{\theta}} = -\theta'. \qquad (2.\ 3)$$

[2]See also [18] for a supersymmetric extension.

[3]The other possibility at hand is to explicitly compute the partition function of the dual theory à la Gawedzki and Kupiainen [19].

The generating functional of this transformation is of type I and it is given by:

$$F = \frac{1}{2} \int_{D, \partial D = S^1} d\tilde{\theta} \wedge d\theta = \frac{1}{2} \oint_{S^1} (\theta' \tilde{\theta} - \theta \tilde{\theta}') d\sigma. \tag{2. 4}$$

From here it is possible to learn about the multivaluedness and periods of the dual variables. In the path integral the canonical transformation is implemented by [12]:

$$\psi_k[\tilde{\theta}(\sigma)] = N(k) \int \mathcal{D}\theta(\sigma) e^{iF[\tilde{\theta}, \theta(\sigma)]} \phi_k[\theta(\sigma)], \tag{2. 5}$$

where $\psi_k[\tilde{\theta}]$ and $\phi_k[\theta]$ are usually chosen as eigenstates corresponding to the same eigenvalue of the respective Hamiltonians and $N(k)$ is a normalization factor. Since θ is periodic, $\phi_k(\theta + a) = \phi_k(\theta)$ implies for $\tilde{\theta}$: $\tilde{\theta}(\sigma + 2\pi) - \tilde{\theta}(\sigma) = 4\pi/a$, which means that $\tilde{\theta}$ must live in the dual lattice of θ.

This makes the duality transformation very simple conceptually, and it also tells us how it can be applied to arbitrary genus Riemann surfaces, because the state $\phi_k[\theta(\sigma)]$ could be the state obtained from integrating the original theory on an arbitrary Riemann surface with boundary. It is also clear that the arguments generalize straightforwardly when we have several commuting isometries.

Let us now consider the principal chiral model defined by the Lagrangian:

$$L = -Tr(g^{-1}\partial_+ g g^{-1} \partial_- g) \tag{2. 6}$$

where g belongs to an arbitrary compact Lie group G. (2. 6) is invariant under $g \to h_1 g h_2$, with $g, h_1, h_2 \in G$. In [5] the general form for the non-abelian dual with respect to the action of the whole group on the left was shown to be:

$$\tilde{L} = \partial_+ \chi^i M_{ij}^{-1} \partial_- \chi^j, \tag{2. 7}$$

where M^{ij} is defined by:

$$M^{ij} = \delta^{ij} + f^{ij}{}_k \chi^k \tag{2. 8}$$

and $f^{ij}{}_k$ are the structure constants in a given basis $\{T^k\}_{k=1}^{\dim G}$ for the Lie algebra $\underline{g}$ of G, normalized so that $Tr(T^k T^l) = -\delta^{kl}$.

(2. 7) is invariant under

$$\chi^i \to R^i{}_j \chi^j, \tag{2. 9}$$

with R in the adjoint representation of $\underline{g}$.

Let us now carry out the canonical transformation approach. The canonical coordinates we are going to use are some arbitrary θ^a, $a = 1, \ldots, \dim G$, living in the group manifold [14]. We can define a matrix $\Omega_a^k(\theta)$ by:

$$T^k \Omega_a^k(\theta) \equiv \frac{\partial g}{\partial \theta^a} g^{-1}. \tag{2. 10}$$

Then $\Omega \equiv T^k \Omega_a^k d\theta^a$ is the Maurer-Cartan form on G.

In these variables L reads:

$$L = \Omega^k_a \Omega^k_b \partial_+ \theta^a \partial_- \theta^b = \Omega^k_a \Omega^k_b (\dot\theta^a \dot\theta^b - \theta'^{\,a} \theta'^{\,b}), \tag{2. 11}$$

where sums over repeated indices are understood.

The canonical momenta are given by:

$$\Pi_a = \frac{\delta L}{\delta \dot\theta^a} = 2\Omega^k_a \Omega^k_b \dot\theta^b \tag{2. 12}$$

and the Hamiltonian:

$$H = \frac{1}{4} \omega^{ak} \omega^{bk} \Pi_a \Pi_b + \Omega^k_a \Omega^k_b \theta'^{\,a} \theta'^{\,b}, \tag{2. 13}$$

where the matrix $\omega^{ak}(\theta)$ is defined from:

$$\omega^{ak}(\theta) \Omega^i_a(\theta) \equiv \delta^{ki}. \tag{2. 14}$$

We can obtain the non-abelian dual of (2. 6) with respect to the left action of the whole group G by performing the canonical transformation generated by:

$$F[\chi, \theta] = -\oint_{S^1} \chi^i J^1_i(\theta) d\sigma, \quad i = 1, \ldots, \mathrm{dim} G, \tag{2. 15}$$

with $J^1_i(\theta)$ the spatial components of the conserved currents associated to the isometry $g \rightarrow hg$, $g, h \in G$, of the initial model. Note that this is a type I generating functional, as in the abelian case, and that in the particular case $g \in U(1)$ it reduces to (2. 4).

This generating functional produces the canonical transformation:

$$\begin{aligned} \tilde\Pi_i &= \frac{\delta F}{\delta \chi^i} = -J^1_i(\theta) \\ \Pi_a &= -\frac{\delta F}{\delta \theta^a}, \end{aligned} \tag{2. 16}$$

from $\{\theta^a, \Pi_a\}$ to $\{\chi^i, \tilde\Pi_i\}$.

As we are going to show the first equation in (2. 16) reflects the equality of the spatial components of the conserved currents in the initial and dual theories. The dual model is invariant under (2. 9). The conserved currents associated to:

$$\delta \chi^i = f^i{}_{jk} \chi^j \omega^k, \tag{2. 17}$$

with ω^k constant parameters, are given by:

$$\tilde J^\mu_i(\chi) = \frac{\delta \tilde L}{\delta(\partial_\mu \omega^i)}. \tag{2. 18}$$

The spatial components are then:

$$\tilde J^1_i(\chi) = 2f^k{}_{ji} \chi^j (M^{-1}_{[kl]} \dot\chi^l - M^{-1}_{(kl)} \chi'^{\,l}), \tag{2. 19}$$

where $M^{-1}_{[kl]} = \frac{1}{2}(M^{-1}_{kl} - M^{-1}_{lk})$ and $M^{-1}_{(kl)} = \frac{1}{2}(M^{-1}_{kl} + M^{-1}_{lk})$.

On the other hand, the canonical momenta derived from (2. 7) are given by:

$$\tilde{\Pi}_i = \frac{\delta \tilde{L}}{\delta \dot{\chi}^i} = 2(M^{-1}_{(ij)}\dot{\chi}^j - M^{-1}_{[ij]}\chi'^{\,j}). \tag{2. 20}$$

Then it can be seen that if we work with the following dual currents[4] [16]:

$$\tilde{I}^\mu_i = \tilde{J}^\mu_i + 2\epsilon^{\mu\nu}\partial_\nu \chi^i \tag{2. 21}$$

differing from $\tilde{J}^\mu_i$ in a total derivative term, we have:

$$\tilde{I}^1_i = -\tilde{\Pi}_i, \tag{2. 22}$$

so that the first equation in (2. 16) is in fact:

$$\tilde{I}^1_i(\chi) = J^1_i(\theta). \tag{2. 23}$$

The conserved currents associated to the invariance $\delta g = \epsilon^k T_k g$, with ϵ^k constant parameters, of the initial theory are:

$$J^\mu_i = \frac{\delta L}{\delta(\partial_\mu \epsilon^i)} = 2\Omega^i_a \partial^\mu \theta^a \tag{2. 24}$$

so that (2. 16) reads:

$$\begin{aligned} \tilde{\Pi}_i &= 2\Omega^i_a \theta'^{\,a} \\ \Pi_a &= 2(\Omega^i_a \chi'^{\,i} - f_{ijk}\chi^i \Omega^j_b \Omega^k_a \theta'^{\,b}). \end{aligned} \tag{2. 25}$$

In order to obtain the dual Hamiltonian we have to express $H(\theta^a, \Pi_a)$ as a function of $\{\chi^i, \tilde{\Pi}_i\}$ by means of the relations (2. 25). This does not seem obvious to do since even in the second equation in (2. 25) Π_a appear as functions of the initial $\{\theta^a\}$ variables. However, in (2. 13) only the combinations $\omega^{ak}\Pi_a$, $\Omega^k_a \theta'^{\,a}$ appear and it is easy to see from (2. 25) that these are given by:

$$\begin{aligned} \omega^{ak}\Pi_a &= 2\chi'^{\,k} - \chi^i f_i{}^{jk}\tilde{\Pi}_j \\ \Omega^k_a \theta'^{\,a} &= \frac{1}{2}\tilde{\Pi}_k. \end{aligned} \tag{2. 26}$$

From here one can check that the time components of the conserved currents of the initial and dual theories are also identified under (2. 25).

The dual Hamiltonian reads:

$$\tilde{H} = \frac{1}{4}\tilde{\Pi}^2 + \chi'^{\,2} - f^i_{jk}\tilde{\Pi}_i \chi'^{\,j}\chi^k + \frac{1}{4} f^{jk}_i f^{mk}_l \chi^i \chi^l \tilde{\Pi}_j \tilde{\Pi}_m, \tag{2. 27}$$

the corresponding Lagrangian being (2. 7).

[4]These are the curvature-free currents [16], i.e. such that $\epsilon_{\mu\nu}(\partial^\mu \tilde{I}^\nu_i - \frac{1}{4} f^{ijk} \tilde{I}^\mu_j \tilde{I}^\nu_k) = 0$, to be compared to the initial J^μ_i, also curvature free.

In the particular case $G = SU(2)$:

$$\tilde{H}_{SU(2)} = \frac{1}{4}\tilde{\Pi}^2 + (\chi')^2 + \sqrt{2}\epsilon^{ijk}\chi'^{\,k}\chi^i\tilde{\Pi}_j + \frac{1}{2}\chi^2\tilde{\Pi}^2 - \frac{1}{2}(\chi\tilde{\Pi})^2, \tag{2. 28}$$

which is the expression obtained in [16] up to normalization.

We can now make some comments concerning our derivation of the canonical transformation.

A simple consequence of our arguments is that they can readily be extended to the action on the left of a subgroup $H \subset G$. One just have to orthogonally decompose the Lie algebra of G, $\underline{g} = \underline{h} \oplus \underline{k}$, where $[\underline{h}, \underline{k}] \subset \underline{k}$, and write:

$$L[g] = L[h] + L[l], \tag{2. 29}$$

with $g = lh$, $h \in H$ and $dll^{-1} \in \underline{k}$. Then the same arguments can be followed for $L[h]$.

The key point which allowed us to find the coupling between the original and dual theories was the fact that in the particular case of principal chiral models the conserved currents of the dual models are given in terms of the canonical variables by just the conjugate momenta themselves[5]. Then both theories can be very easily related. Of course, in order to do this we have used some explicit information we had a priori, which is the fact that the non-abelian duals of the principal chiral models of group G with respect to the action of the whole group on the left had already been studied in the literature [5, 17]. There is one feature that is especial for principal chiral models. This is the fact that as the original theory is invariant under $g \to h_1 g h_2$, with $g, h_1, h_2 \in G$, and only the left action is used in order to construct the dual the other isometry, commuting with the one acting on the left, still remains, so that there are some conserved currents in the dual model. Then it makes sense to try to couple both theories by means of their conserved currents, especially because the dual ones are so easily expressed as functions of the canonical variables. This fact makes the extension of the explicit construction we have just performed to arbitrary σ- models not obvious, since in general the dual model will not have conserved currents at all after a non-abelian duality transformation is made. This is what happens for instance in the case of WZW models, where the only non-anomalous action that can be gauged is the vectorial action, which yields dual models without conserved currents (at least locally). The generalization to these models remains then an interesting open problem which we hope to address in a future publication.

The generating functional (2. 15) is linear in the dual variables but not in the original ones, so we expect a priori that it will receive quantum corrections when implemented in the path integral [12]:

$$\psi_k[\chi^i(\sigma)] = N(k)\int \prod_{a=1}^{\dim G} \mathcal{D}\theta^a(\sigma) e^{iF[\chi,\theta(\sigma)]}\phi_k[\theta^a(\sigma)]. \tag{2. 30}$$

This formula is the starting point to work out the multivaluedness and periods of the dual variables. This information is not available at present for general non-abelian duality transformations, as we mentioned in the introduction. Only in the particular case of σ-models with chiral currents and as a consequence of the Polyakov-Wiegmann property [15] satisfied

[5] This was first noticed in [16] for the case of an $SU(2)$ principal chiral model.

by WZW models we know the space in which the dual variables live [7]. Within the approach of the canonical transformation one needs to compute the quantum corrections to the generating functional (2. 15) in order to obtain this information.

It is easy to check that the following relation holds[6]:

$$\tilde{I}^i_\mu[\chi]\psi_k[\chi^i(\sigma)] = N(k) \int \prod_{a=1}^{\dim G} \mathcal{D}\theta^a(\sigma) e^{iF[\chi,\theta]} J^i_\mu[\theta]\phi_k[\theta^a(\sigma)], \tag{2. 31}$$

with $\tilde{I}^i_\mu$ given by (2. 21). So if we choose $\psi_k[\chi]$ and $\phi_k[\theta]$ as eigenfunctions of the respective conserved currents with the same eigenvalue, (2. 30) holds with F given by the classical expression (2. 15). This implies that although the generating functional is not linear in $\{\theta^a\}$ it is in fact exact to all orders in α'. However this argument is formal and in order to further establish that F is generating a quantum transformation one would need to take into account renormalization effects.

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[6]This was first noticed in [16] for $G = SU(2)$.

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